• Giovanni- H-convergence:

- $\Omega \subseteq \mathbb{R}^N$ open set, $\alpha, \alpha', \beta, \beta' > 0$ s.t.

$$\begin{cases} 0 < \alpha < \beta < +\infty \\ 0 < \alpha' < \beta' < +\infty \end{cases}$$

- $M(\alpha, \beta, \Omega) := \{ A \in [L^\infty(\Omega)]^{N^2} : (A(x)\lambda, \lambda) \geq \alpha |\lambda|^2, \ |A(x)\lambda| \leq \beta |\lambda| \}$

- Def: $(A^\varepsilon)_\varepsilon \in M(\alpha, \beta, \Omega)$ **H-converges** to $A^0 \in M(\alpha', \beta', \Omega)$, $A^\varepsilon \xrightarrow{H} A^0$, if $\forall \omega \subset\subset \Omega$ and $f \in H^{-1}(\omega)$, the solution u^ε of

$$\begin{cases} -\operatorname{div}(A^\varepsilon \nabla u^\varepsilon) = f & \text{in } \omega, \\ u^\varepsilon \in H^1_0(\omega), \end{cases}$$

is s.t.

$$\begin{cases} u^\varepsilon \xrightarrow{w\text{-}H^1_0(\omega)} u^0, \\ A^\varepsilon \nabla u^\varepsilon \xrightarrow{[L^2(\omega)]^{N^2}} A^0 \nabla u^0; \end{cases}$$

where u_0 solves

$$\begin{cases} -\operatorname{div}(A^0 \nabla u^0) = f & \text{in } \omega, \\ u^0 \in H^1_0(\omega). \end{cases}$$

- Examples: i) $A^\varepsilon \to A^0$ a.e. $\Rightarrow A^\varepsilon \xrightarrow{H} A^0$

  ii) if $N=1$: $A^\varepsilon \xrightarrow{H} A^0 \Leftrightarrow \frac{1}{A^\varepsilon} \xrightarrow{w^*L^\infty(\Omega)} \frac{1}{A^0}$

- Properties of the H-convergence:

  - Locality: i) $(A^\varepsilon)_\varepsilon$ has at most one H-limit

    ii) if $(A^\varepsilon)_\varepsilon, (B^\varepsilon)_\varepsilon \in M(\alpha,\beta,\Omega)$ s.t.

    $$\begin{cases} A^\varepsilon \xrightarrow{H} A^0, \\ B^\varepsilon \xrightarrow{H} B^0, \end{cases}$$

    and $A^\varepsilon = B^\varepsilon$ in $\omega \subset \Omega$, then

    $A^0 = B^0$ in ω.

  - Convergence of the energy & irrelevance of the boundary cond.:

Thm:

Let $(A^\varepsilon)_\varepsilon \in M(\alpha,\beta,\Omega)$ s.t. $A^\varepsilon \xrightarrow{H} A^0 \in M(\alpha',\beta',\Omega)$.

Assume:

$$\begin{cases} u^\varepsilon \in H^1(\Omega) \\ f^\varepsilon \in H^{-1}(\Omega) \\ -\operatorname{div}(A^\varepsilon \nabla u^\varepsilon) = f^\varepsilon \text{ in } \Omega \\ u^\varepsilon \to u^0 \quad w\text{-}H^1(\Omega) \\ f^\varepsilon \to f^0 \quad s\text{-}H^{-1}(\Omega) \end{cases}$$

Then:

- $A^\varepsilon \nabla u^\varepsilon \to A^0 \nabla u^0 \quad w\text{-}[L^2(\Omega)]^N$
- $(A^\varepsilon \nabla u^\varepsilon, \nabla u^\varepsilon) \xrightarrow{w^*\mathcal{D}'} (A^0\nabla u^0, \nabla u^0)$

$\Big\{$ for "compact sets", something can be lost at the boundary

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In order to prove the above thm, we need the following results:

- **Lemma 1**: [the div-curl lemma]

Let $\xi^\varepsilon, v^\varepsilon$ be s.t.

$$\begin{cases} \xi^\varepsilon \in [L^2(\Omega)]^N \\ \xi^\varepsilon \rightharpoonup \xi^0 \text{ w-}[L^2(\Omega)]^N \\ \operatorname{div} \xi^\varepsilon \longrightarrow \operatorname{div} \xi^0 \ H^{-1}(\Omega) \\ v^\varepsilon \in H^1(\Omega) \\ v^\varepsilon \rightharpoonup v^0 \text{ w-}H^1(\Omega) \end{cases}$$

Then:

$$(\xi^\varepsilon, \nabla v^\varepsilon) \xrightarrow{w\text{-}\mathcal{D}'} (\xi^0, \nabla v^0).$$

Proof [sketch]

Let $\varphi \in C_c^\infty(\Omega)$; then:

$$\int_\Omega (\xi^\varepsilon, \nabla v^\varepsilon)\varphi = -\underbrace{\langle \operatorname{div} \xi^\varepsilon, v^\varepsilon \varphi \rangle_{H^{-1} \times H^1_0}}_{} \underbrace{- \int_\Omega (\xi^\varepsilon, \nabla \varphi) v^\varepsilon}_{}$$

$$\longrightarrow \ldots \qquad \longrightarrow \ldots$$

$$= \int_\Omega (\xi^0, \nabla v^0)\varphi$$

$\boxtimes$

- <u>Lemma 2</u>:

Let $(A^\varepsilon)_\varepsilon \subseteq M(\alpha, \beta, \Omega)$ and assume

$$\begin{cases} u^\varepsilon \in H^1(\Omega) \\ \text{(i)} \quad u^\varepsilon \rightharpoonup u^0 \ \text{w-}H^1 \\ \text{(ii)} \quad \xi^\varepsilon := A^\varepsilon \nabla u^\varepsilon \xrightarrow{w\text{-}L^2} \xi^0 \in [L^2(\Omega)]^N \\ \text{(iii)} \quad -\operatorname{div} \xi^\varepsilon \longrightarrow -\operatorname{div} \xi^0 \ \text{s-}H^{-1}(\Omega) \\ \text{(iv)} \quad v^\varepsilon \in H^1(\Omega) \\ \text{(v)} \quad v^\varepsilon \rightharpoonup v^0 \ \text{w-}H^1 \\ \text{(vi)} \quad \eta^\varepsilon := {}^tA^\varepsilon \nabla v^\varepsilon \rightharpoonup \eta^0 \ \text{w-}[L^2]^N \\ \text{(vii)} \quad -\operatorname{div} \eta^\varepsilon \longrightarrow -\operatorname{div} \eta^0 \ \text{s-}H^{-1}(\Omega) \end{cases}$$

Then:

$$(\xi^0, \nabla v^0) = (\nabla u^0, \eta^0) \quad \text{a.e. in } \Omega.$$

<u>Proof</u>:

use lemma 1. $\boxed{}$

· Prop.:

$$A^\varepsilon \xrightarrow{H} A^0 \Longrightarrow {}^tA^\varepsilon \xrightarrow{H} {}^tA^0.$$

[Proof of Thm 1]

Let $\omega \subset\subset \Omega$, $\xi^\varepsilon := A^\varepsilon \nabla u^\varepsilon$.

· Then $(\xi^\varepsilon)_\varepsilon$ is bdd in $L^2 \Rightarrow \exists\, \xi^0$ s.t. $\xi^\varepsilon \rightharpoonup \xi^0$ w-L^2. [(ii)]

· Notice that:

$$\int \xi^\varepsilon \nabla \varphi = \int f^\varepsilon \varphi \longrightarrow \int f^0 \varphi$$

$$\downarrow$$

$$\int \xi^0 \nabla \varphi \qquad \Downarrow \; -\operatorname{div} \xi^0 = f^0 \quad \text{[(iii)]}$$

· Let $f \in H^{-1}(\omega)$; let v^ε s.t.

$$\begin{cases} -\operatorname{div}({}^tA^\varepsilon \nabla v^\varepsilon) = f = -\operatorname{div}({}^tA^0 \nabla v^0) \\ v^\varepsilon \in H^1_0(\Omega) \end{cases}$$

$$\Downarrow \; {}^tA^\varepsilon \xrightarrow{H} {}^tA^0$$

$$\begin{cases} v^\varepsilon \rightharpoonup v^0 & \text{w-}H^1 \\ {}^tA^\varepsilon \nabla v^\varepsilon \longrightarrow {}^tA^0 \nabla v^0 & \text{w-}L^2 \end{cases} \quad \text{[(vii)]}$$

· $(\xi^0, \nabla v^0) = (\nabla u^0, {}^tA^0 \nabla v^0)$ a.e. in Ω

this can be any $\lambda \in \mathbb{R}^N$, since f is arbitrary $\Rightarrow$ $\boxed{\xi^0 = A^0 \nabla u^0}$ a.e. in Ω

• by Lemma 1 we get also the second convergence

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• <u>Compactness of H-convergence</u>:

Let $(A^\varepsilon)_\varepsilon \in M(\alpha, \beta, \Omega)$.

Then $\exists\ (A^{\varepsilon_k})_k$, $A^0 \in M(\alpha, \beta^2/\alpha, \Omega)$

s.t.

$$A^{\varepsilon_k} \xrightarrow{H} A^0.$$

• <u>Corrector matrix</u>:

Let $\omega \subset\subset \Omega$ and define w_1^ε s.t.

$$\begin{cases} w_1^\varepsilon \in H^1(\omega) \\ w_1^\varepsilon \rightharpoonup (\lambda, x) \text{ in } H^1(\omega) \\ -\operatorname{div}(A^\varepsilon \nabla w_1^\varepsilon) \to \operatorname{div}(A^0 \lambda) \quad H^{-1}(\omega) \end{cases}$$

[$\exists$ of w_1^ε: $\begin{cases} -\operatorname{div}(A^\varepsilon \nabla w_1^\varepsilon) = -\operatorname{div}(A^0 \nabla((\lambda, x)\varphi(x)) \text{ in } \omega_2 \\ w_1^\varepsilon \in H_0^1(\omega_2) \end{cases}$

$\varphi \in C_c^\infty(\omega_2)$, $\varphi \equiv 1$ on ω

$\omega \subset\subset \omega_2 \subset\subset \Omega$.

• not !, but all $\rightharpoonup (\lambda, x)$ in the limit]

• Def: the corrector matrix $P^\varepsilon \in [L^2(\Omega)]^{N^2}$ is defined by:

$$P^\varepsilon \lambda := \nabla w^\varepsilon_\lambda$$

• Properties: i) $P^\varepsilon \rightharpoonup Id \quad w\text{-}(L^2)^{N^2}$

ii) $A^\varepsilon P^\varepsilon \rightharpoonup A^0 \quad w\text{-}(L^2)^{N^2}$

iii) ${}^tP^\varepsilon A^\varepsilon P^\varepsilon \xrightarrow{w\text{-}\mathcal{D}'} A^0$

• Thm: (strong approximation)

Assume: $\begin{cases} u^\varepsilon \in H^1(\Omega), \ f^\varepsilon \in H^{-1}(\omega) \\ -\operatorname{div}(A^\varepsilon \nabla u^\varepsilon) = f^\varepsilon \text{ in } \omega \\ u^\varepsilon \rightharpoonup u^0 \quad w\text{-}H^1(\omega) \\ f^\varepsilon \to f^0 \quad s\text{-}H^{-1} \end{cases}$

Then: $\boxed{\nabla u^\varepsilon = P^\varepsilon \nabla u^0 + z^\varepsilon}$

where $z^\varepsilon \to 0 \quad s\text{-}[L^1_{loc}(\omega)]^N$

Furthermore, if $\begin{cases} \|P^\varepsilon\|_{L^q} \le C, \ 2 \le q \le +\infty, \\ \nabla u^0 \in L^p, \end{cases}$

then $z^\varepsilon \to 0 \ s\text{-}[L^r_{loc}]^N, \ \frac{1}{r} = \max\left\{\frac{1}{2}, \frac{1}{p} + \frac{1}{q}\right\}$.

Finally, if

$$\int_\omega (A^\varepsilon \nabla u^\varepsilon, \nabla u^\varepsilon) \longrightarrow \int (A^0 \nabla u^0, \nabla u^0),$$

then $\quad z^\varepsilon \longrightarrow 0 \quad s\text{-}[L^2(\omega)]^N.$